

Genuine Entanglement of Four Qubit Cluster Diagonal States

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Abstract

We reduce the necessary and sufficient biseparable conditions of the four qubit cluster diagonal state to concise forms. Only 4 out of the 15 parameters are proved to be relevant in specifying the genuine entanglement of the state. Using the relative entropy of entanglement as the entanglement measure, we analytically find the genuine entanglement of all the four qubit cluster diagonal states. The formulas of the genuine entanglement are of five kinds, for seven different parameter regions of entanglement.

1 Introduction

In quantum information and quantum many-body physics, multipartite entanglement is still a phenomenon that is poorly understood [1] [2]. Experimentally, multipartite entanglement has been observed in ion traps [3], photon polarization [4], superconducting phase or circuit qubit systems [5], and nitrogen-vacancy centers in diamond [6]. More than ten qubits have been entangled in Greenberger-Horne-Zeilinger (GHZ) state in ion traps [3], and six-photon cluster state has been observed [7]. Usually the multipartite entangled states observed in the experiments are graph states. Graph states are based on graphs and are very useful in constructing quantum error-correcting codes. Due to noise and imperfections in preparation, the states prepared are usually mixed states. Typically they are the so called graph diagonal states. Theoretical research has been concentrated on the separability and biseparability of the prepared states [8] [9] [10] [11] [12] [13]. Cluster states are special graph states. Recently, necessary and sufficient biseparable criterion for four qubit cluster diagonal states was obtained [14]. Biseparable states are multipartite quantum states that can be expressed as a convex sum of the projectors of product vectors and bipartite entangled vectors [15]. Hence a genuine multipartite entangled state is not biseparable. Full separable states are those that can be expressed as a convex sum of the projectors of product vectors. So full separable state set is the subset of biseparable state set. The criterion for full separability of a four qubit cluster state is not known. Thus the quantification of the entanglement is not available when the entanglement measure involves with the full separable state set.

Quantifying multipartite entanglement is a difficult

problem even for a pure multipartite state. Measures such as entanglement cost, distillable entanglement work well and have clear operational meanings in bipartite systems. However, it is not easy to extend them to multipartite systems. The relative entropy of entanglement (REE) is a valid measure for multipartite as well as for bipartite systems[16]. For a given quantum state σ , REE is defined as $E = \min_{\rho \in Sep} S(\sigma \parallel \rho)$, where Sep is the separable state set, $S(\sigma \parallel \rho) = Tr(\sigma \log_2 \sigma - \sigma \log_2 \rho)$ is the relative entropy. In multipartite system, genuine entanglement measured by relative entropy can be defined by minimizing the relative entropy over all biseparable states. Vedral *et al.* [16] had proposed such a definition of entanglement for three-partite system. In this paper, based on the biseparable criterion, we study the genuine entanglement of four qubit cluster diagonal states measured by REE.

2 Cluster diagonal state and its necessary and sufficient biseparable criteria

A simple graph $G = (V, \Gamma)$ is composed of a set V of n vertices and a set of edges characterized by the adjacency matrix Γ . The $n \times n$ symmetric matrix Γ has nullified diagonal elements and $\Gamma_{ij} = 1$ if vertices i and j are connected and $\Gamma_{ij} = 0$ otherwise. The neighbourhood of a vertex i is denoted by $N_i = \{j \in V | \Gamma_{ij} = 1\}$. Consider a system of n qubits, and define the mutually commuting stabilizer operators:

$$K_i = X_i \prod_{j \in N_i} Z_j \quad (1)$$

where X_i and Z_j are Pauli X and Z matrices at vertices i and j , respectively. The operators stabilize the graph state

$$|G\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mu=0}^1 (-1)^{\frac{1}{2}\mu\Gamma\mu^T} |\mu\rangle \quad (2)$$

such that $K_i |G\rangle = |G\rangle$. Graph state basis are $|G_{\alpha_1 \alpha_2 \dots \alpha_n}\rangle = Z_1^{\alpha_1} Z_2^{\alpha_2} \dots Z_n^{\alpha_n} |G\rangle$ with $\alpha_i = 0, 1$, each of them is the common eigenstate of all the operators K_i , with eigenvalues ± 1 .

The graph of the four qubit cluster state $|Cl\rangle$ has three edges such that $\Gamma_{i,i+1} = \Gamma_{i+1,i} = 1$ ($i = 1, 2, 3$) and all

the other elements of Γ are 0. The stabilizer operators are

$$K_1 = X_1 Z_2 I_3 I_4, K_2 = Z_1 X_2 Z_3 I_4, \quad (3)$$

$$K_3 = I_1 Z_2 X_3 Z_4, K_4 = I_1 I_2 Z_3 X_4. \quad (4)$$

where I_i is the identity matrix for vertex i . The four qubit cluster basis states are $|Cl_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}\rangle$ with $K_i |Cl_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}\rangle = (-1)^{\alpha_i} |Cl_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}\rangle$. A four qubit cluster diagonal state σ is the probability mixture of the cluster basis states

$$\sigma = \sum_{\alpha, \beta, \gamma, \delta=0}^1 F_{\alpha\beta\gamma\delta} |Cl_{\alpha\beta\gamma\delta}\rangle \langle Cl_{\alpha\beta\gamma\delta}|, \quad (5)$$

where $F_{\alpha\beta\gamma\delta} \geq 0$ and $\sum_{\alpha, \beta, \gamma, \delta=0}^1 F_{\alpha\beta\gamma\delta} = 1$.

The original necessary and sufficient biseparable criteria are [14]

$$2F_{\alpha\beta\gamma\delta} \leq \sum_{\xi, \eta=0}^1 (F_{\alpha\xi\eta\delta} + F_{\alpha\xi\eta\bar{\delta}} + F_{\bar{\alpha}\xi\eta\delta}) \quad (6)$$

$$2F_{\alpha\beta\gamma\delta} + 2F_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \leq \sum_{\xi, \eta=0}^1 (F_{\alpha\xi\eta\delta} + F_{\alpha\xi\eta\bar{\delta}} + F_{\bar{\alpha}\xi\eta\delta} + F_{\bar{\alpha}\xi\eta\bar{\delta}}) \quad (7)$$

for all the subscripts $\alpha, \beta, \gamma, \delta, \mu, \nu = 0, 1$, where $F_{\alpha\beta\gamma\delta} = \langle C_{4\alpha\beta\gamma\delta} | \sigma | C_{4\alpha\beta\gamma\delta} \rangle$. Violation of the biseparable criteria means genuine entanglement. Note that the right hand side of inequality (7) is just equal to 1 according to the normalization of σ in cluster state basis. Hence the two criteria can be written as

$$2F_{\alpha\beta\gamma\delta} + \sum_{\xi, \eta=0}^1 F_{\bar{\alpha}\xi\eta\bar{\delta}} \leq 1, \quad (8)$$

$$F_{\alpha\beta\gamma\delta} + F_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \leq \frac{1}{2}, \quad (9)$$

Denote

$$p_{2\alpha+\delta} = \max_{\beta, \gamma} \{F_{\alpha\beta\gamma\delta}\}, \quad (10)$$

$$p_{4+2\alpha+\delta} = \sum_{\beta, \gamma} F_{\alpha\beta\gamma\delta} - p_{2\alpha+\delta}. \quad (11)$$

Then inequalities (8) and (9) can be further reduced to

$$p_0 + p_3 \leq \frac{1}{2}, \quad (12)$$

$$2p_0 + p_3 + p_7 \leq 1, \quad (13)$$

$$2p_3 + p_0 + p_4 \leq 1, \quad (14)$$

$$p_1 + p_2 \leq \frac{1}{2}, \quad (15)$$

$$2p_1 + p_2 + p_6 \leq 1, \quad (16)$$

$$2p_2 + p_1 + p_5 \leq 1, \quad (17)$$

We will prove that inequality (9) and inequalities (12), (15) are equivalent. For any β, γ, μ, ν , if (9) is violated, then one of (12) and (15) should be violated from the

definition of $p_{2\alpha+\delta}$. If (12) and/or (15) are violated, then at least one case of (9) is violated for some β, γ, μ, ν , since (12) and (15) are special cases of (9). The same reasoning can be applied to prove the equivalence of inequality (8) and inequalities (13)(14)(16)(17).

If inequality (12) is violated, then inequalities (15)-(17) are preserved due to the normalization $\sum_{\alpha, \beta, \gamma, \delta} F_{\alpha\beta\gamma\delta} = \sum_{\alpha, \delta} (p_{2\alpha+\delta} + p_{4+2\alpha+\delta}) = 1$. If inequality (13) is violated, we also have inequalities (15)-(17) been fulfilled. We conclude that if one of the inequalities (12)-(14) is violated, then inequalities (15)-(17) are all preserved and vice versa. So we only need to consider half of the inequalities been violated. We consider the violation of inequalities (12)-(14), in the following we will mainly work on the parameters p_0, p_3, p_4, p_7 . Suppose, for example, the maximal of $F_{0\beta\gamma 0}$ ($\beta, \gamma = 0, 1$) be F_{0000} , the maximal of $F_{1\beta\gamma 1}$ ($\beta, \gamma = 0, 1$) be F_{1001} , then $p_0 = F_{0000}$, $p_3 = F_{1001}$, $p_4 = F_{0010} + F_{0100} + F_{0110}$, $p_7 = F_{1011} + F_{1101} + F_{1111}$.

3 Entanglement measure

The (genuine) REE [16] defined for a tripartite entangled state can easily be extended to a generic multipartite state. The genuine entanglement of a genuine entangled state σ measured by the REE is

$$E = \min_{\rho \in \text{Bisep}} S(\sigma \| \rho) = -\text{Tr}(\sigma \log_2 \sigma - \sigma \log_2 \varrho), \quad (18)$$

where Bisep is the biseparable set, ϱ is the closest biseparable state, namely, biseparable state that minimizes the relative entropy. The genuine REE measures how ‘far’ is the genuine entangled state from its nearest biseparable state, i.e., the state that is not genuine entangled. For a cluster diagonal state σ , it is easy to show that ϱ is also a cluster diagonal state following the reasoning of declaration that the closest separable state of a Bell diagonal entangled state is a Bell diagonal state [16]. Let $\varrho = \sum_{\alpha, \beta, \gamma, \delta=0}^1 \Lambda_{\alpha\beta\gamma\delta} |Cl_{\alpha\beta\gamma\delta}\rangle \langle Cl_{\alpha\beta\gamma\delta}|$, with $\{\Lambda_{\alpha\beta\gamma\delta}\}$ forming a probability distribution. The genuine REE is

$$E = \sum_{\alpha, \beta, \gamma, \delta=0}^1 F_{\alpha\beta\gamma\delta} \log_2 \frac{F_{\alpha\beta\gamma\delta}}{\Lambda_{\alpha\beta\gamma\delta}}. \quad (19)$$

The convex property of $-\log$ function means that the closest biseparable state ϱ should be at the boundary of the biseparable state set. Denote $\lambda_{2\alpha+\delta} = \max_{\beta, \gamma} \{\Lambda_{\alpha\beta\gamma\delta}\}$, $\lambda_{4+2\alpha+\delta} = \sum_{\beta, \gamma} \Lambda_{\alpha\beta\gamma\delta} - \lambda_{2\alpha+\delta}$, then at least one of the equality should be reached in the biseparable criteria of the state ϱ , namely, if we replace p_i with λ_i for inequalities (12)-(17), then at least one of the inequalities should be an equation. We use p_i to specify the possibly genuine entangled state σ , use λ_i to specify the closest biseparable state ϱ .

4 Three parameter states

Suppose a genuine entangled state σ violate inequality (12) or inequality (13) or both of them, while leaving

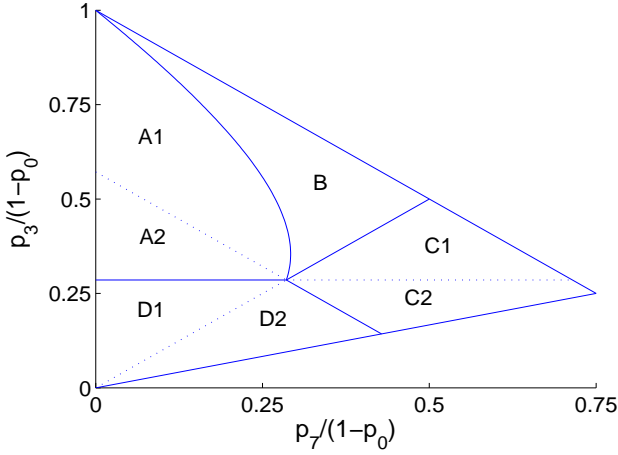


Figure 1: (Color online) Edge lines or curve and genuine entanglement or biseparable regions for $p_0 = 0.3$. Genuine entangled state Regions: A_1 violates (12) and (13), A_2 violates (12), the genuine REE can be $E_A, E_{A''}$ or $E_{A'''}$ due to the fourth parameter p_4 ; B violates (12) and (13), the genuine REE is E_B ; C_1 violates (12) and (13), C_2 violates (13), the genuine REE is E_C . Biseparable state region: D_2 . Region whose biseparability yet to be determined by the fourth parameter: D_1 . In regions D_1, D_2 , both (12) and (13) are preserved. For all the regions, possible violation of (14) will be discussed in the next section.

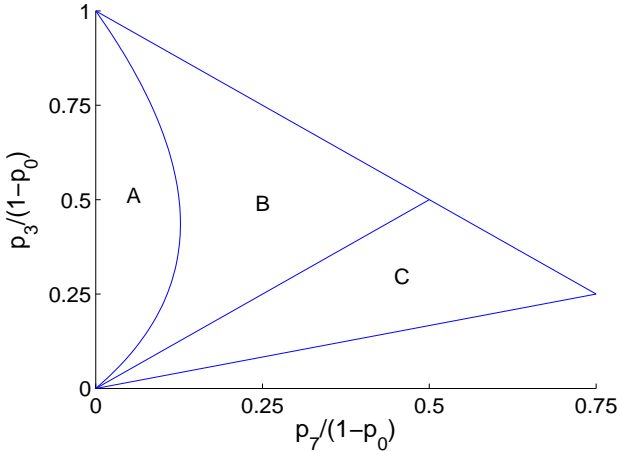


Figure 2: (Color online) Edge lines or curve of genuine entanglement regions for $p_0 = 0.6$. Genuine entangled state Regions: A, B, C , both (12) and (13) are violated. The genuine REE for region A can be $E_A, E_{A''}$ or $E_{A'''}$ due to the fourth parameter p_4 , The genuine REE is E_B for B , is E_C for C .

inequality (14) pending for further consideration in the next section. Hence σ is characterized by three parameters p_0, p_3, p_7 concerning its genuine entanglement property. Without considering the fourth parameter p_4 or possible violation of inequality (14), the genuine entanglement obtained in this section is only the candidate of the final result.

When inequality (12) is violated, namely, $p_0 + p_3 > \frac{1}{2}$, the parameter regions can be shown as A_1, A_2, B and C_1 in Fig.1 for $p_0 < \frac{1}{2}$ or A, B, C in Fig.2 for $p_0 \geq \frac{1}{2}$. When inequality (13) is violated, namely, $2p_0 + p_3 + p_7 > 1$, the parameter regions can be shown as A_2, B, C_1 and C_2 in Fig.1 for $p_0 < \frac{1}{2}$ or A, B, C in Fig.2 for $p_0 \geq \frac{1}{2}$.

We first consider the case of $p_0 < \frac{1}{2}$. For any give genuine entangled state in regions A_1, A_2, B, C_1 and C_2 , there are three classes of possible closest biseparable states.

Class I is the closest biseparable state set with

$$\lambda_0 + \lambda_3 = \frac{1}{2}. \quad (20)$$

Using Lagrange multiplier to minimize the relative entropy, we can determine

$$\lambda_0 = \frac{p_0}{2(p_0 + p_3)}, \quad \lambda_3 = \frac{p_3}{2(p_0 + p_3)} \quad (21)$$

$$\Lambda_{\alpha\beta\gamma\delta} = \frac{F_{\alpha\beta\gamma\delta}}{2(1 - p_0 - p_3)}, \quad \text{for all the others} \quad (22)$$

and the genuine REE is

$$E_A = 1 - H_2(p_0 + p_3). \quad (23)$$

where $H_2(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function.

Class II is the closest biseparable state set with

$$2\lambda_0 + \lambda_3 + \lambda_7 = 1. \quad (24)$$

We obtain the solution

$$\lambda_0 = \frac{1}{2}(1 - p_3 - p_7), \quad \lambda_3 = p_3, \quad \lambda_7 = p_7, \quad (25)$$

$$\Lambda_{\alpha\beta\gamma\delta} = \frac{(1 - p_3 - p_7)F_{\alpha\beta\gamma\delta}}{2(1 - p_0 - p_3 - p_7)}, \quad \text{for all the others} \quad (26)$$

and the genuine REE

$$E_C = (1 - p_3 - p_7)[1 - H_2(\frac{p_0}{1 - p_3 - p_7})]. \quad (27)$$

Class III is the closest biseparable state set with both (20) and (24). The Lagrange multiplier solution of minimizing (19) with restrictions of (20) and (24) and the normalization condition is

$$\lambda_0 = \frac{1}{2}(1 - p_3 - p_7), \quad \lambda_3 = \lambda_7 = \frac{1}{2}(p_3 + p_7) \quad (28)$$

$$\Lambda_{\alpha\beta\gamma\delta} = \frac{(1 - p_3 - p_7)F_{\alpha\beta\gamma\delta}}{2(1 - p_0 - p_3 - p_7)}, \quad \text{for all the others} \quad (29)$$

The genuine REE is

$$E_B = 1 - (p_3 + p_7)H_2(\frac{p_3}{p_3 + p_7}) - (1 - p_3 - p_7)H_2(\frac{p_0}{1 - p_3 - p_7}). \quad (30)$$

For class I closest biseparable states, we should check if $2\lambda_0 + \lambda_3 + \lambda_7 \leq 1$ is preserved. Otherwise, the supposed closest state is not biseparable, the solution (21)-(23) is no longer proper. Using the solution (21)-(22), we have $2\lambda_0 + \lambda_3 + \lambda_7 = \frac{1}{2}(1 + \frac{p_0}{p_0+p_3} + \frac{p_7}{1-p_0-p_3})$, which is less than or equal to 1 when

$$p_7 \leq \frac{1-p_0-p_3}{p_0+p_3}p_3 \equiv p_{AB}. \quad (31)$$

This characterizes the region A (the union A_1 and A_2 in Fig.1). The equality in (31) represents the border curve in Fig.1 or Fig.2 that divides region A and region B .

For class II closest biseparable states, we should check whether $\lambda_0 + \lambda_3 \leq \frac{1}{2}$ is preserved. If it is not, the supposed closest state will go out of the biseparable state set, the solution (25)-(27) is not valid. Using (28), we have $\lambda_0 + \lambda_3 = \frac{1}{2}(1 + p_3 - p_7)$, which is less than or equal to $\frac{1}{2}$ only when

$$p_7 \geq p_3. \quad (32)$$

This characterizes region C (the union C_1 and C_2 in Fig.1). The equality in (32) represents the border line in Fig.1 or Fig.2 that divides region C and region B .

All the states in region A violate (32), thus they have not closest biseparable states of class II. The possible solutions are class I and class III with genuine entanglement E_A and E_B , respectively. Since $1 - H_2(x)$ is a convex function, we have $E_B \geq E_A$, the genuine REE of region A is E_A .

All the states in region C violate (31), so they have not closest biseparable states of class I. The possible solutions are class II and class III with genuine entanglement E_C and E_B , respectively. However, E_C is only a part of E_B , we have $E_B \geq E_C$. Hence, the genuine entanglement of region C is E_C .

The states in region B (the states at the border curve with A and border line with C are not included) violate either (31) or (32). Their closest biseparable states can not be in class I or class II. They are in class III. The genuine REE of region B is E_B .

The case of $p_0 \geq \frac{1}{2}$ can be similarly analyzed. In Fig.2, all the states in regions A, B, C are genuinely entangled.

The conclusion for three parameter states is: the candidate genuine REE in regions A, B and C is E_A, E_B and E_C . Note that p_3 is the maximum of $F_{1\beta\gamma 1}$, so $3p_3 \geq p_7$. Hence the region with $p_3 < \frac{1}{3}p_7$ is meaningless.

The closest biseparable states of the genuine entangled state in region A and C are on the lines of (20) and (24) (hyperplanes in fact), respectively. The closest biseparable states of the genuine entangled state in region B are on the intersection point of the lines (20) and (24).

5 Four parameter states

The regions are classified as A, B, C, D with parameters p_0, p_3, p_7 in last section. We will consider the further classification of each regions by the fourth parameter p_4 . The biseparable state set is shown in Fig.3 for $\lambda_0 = 0.2$. The three classes of closest biseparable state sets defined

in last section are shown in Fig.3 with surfaces I, II and intersection line III. There are two new classes of closest biseparable states appear when the fourth parameter p_4 is considered.

Class IV (shown in Fig.3 with surface IV) is the closest biseparable state set with

$$\lambda_0 + 2\lambda_3 + \lambda_4 = 1. \quad (33)$$

The Lagrange multiplier solution of minimizing (19) with restrictions of (33) is

$$\begin{aligned} \lambda_3 &= \frac{1}{2}(1 - p_0 - p_4), \quad \lambda_0 = p_0, \quad \lambda_4 = p_4, \\ \Lambda_{\alpha\beta\gamma\delta} &= \frac{(1 - p_0 - p_4)F_{\alpha\beta\gamma\delta}}{2(1 - p_0 - p_3 - p_4)}, \text{ for all the others} \end{aligned} \quad (34)$$

The genuine REE is

$$E_{A'''} = (1 - p_0 - p_4)[1 - H_2(\frac{p_3}{1 - p_0 - p_4})]. \quad (36)$$

Class V (shown in Fig.3 with line V) is the closest biseparable state set with both (20) and (33). The solution of (19) is

$$\begin{aligned} \lambda_3 &= \frac{1}{2}(1 - p_0 - p_4), \quad \lambda_0 = \lambda_4 = \frac{1}{2}(p_0 + p_4), \\ \Lambda_{\alpha\beta\gamma\delta} &= \frac{(1 - p_0 - p_4)F_{\alpha\beta\gamma\delta}}{2(1 - p_0 - p_3 - p_4)}, \text{ for all the others} \end{aligned} \quad (37)$$

The genuine REE is

$$\begin{aligned} E_{A''} &= 1 - (p_0 + p_4)H_2(\frac{p_0}{p_0 + p_4}) \\ &\quad - (1 - p_0 - p_4)H_2(\frac{p_3}{1 - p_0 - p_4}). \end{aligned} \quad (39)$$

The genuine entanglement regions in the four parameter system with given p_0 are determined by parameters $\frac{p_3}{1-p_0}, \frac{p_7}{1-p_0}, \frac{p_4}{1-p_0}$. All the genuine entanglement regions described in Fig.1 and Fig.2 now are three dimensional when p_4 is considered. The regions are three dimensional by adding the third dimension p_4 to the two dimensional graphs shown in Fig.1 or Fig.2. For example, the bottom of the three dimensional region A is the graph in Fig.1 or Fig.2. The roof of A is determined by the condition

$$p_0 + p_4 + p_3 + p_7 = 1, \quad (40)$$

which comes from the normalization condition $1 = \sum_{\alpha,\beta,\gamma,\delta} F_{\alpha\beta\gamma\delta} \geq p_0 + p_4 + p_3 + p_7$. The other border surfaces are $p_7 = 0$, $p_7 = p_{AB}$ and $p_0 + p_3 = \frac{1}{2}$.

For region A , we have proven at last section that the closest biseparable states can not be in class II. Suppose the closest biseparable states are in class I, the solution is (21)-(23). Then $\lambda_0 + 2\lambda_3 + \lambda_4 = \frac{1}{2}(1 + \frac{p_3}{p_0+p_3} + \frac{p_4}{1-p_0-p_3})$. The biseparable condition $\lambda_0 + 2\lambda_3 + \lambda_4 \leq 1$ is preserved only when

$$p_4 \leq \frac{1-p_0-p_3}{p_0+p_3}p_0 \equiv p_{A'A''}. \quad (41)$$

We denote the region in A limited by (41) as A' .

For the other part of A , suppose the closest biseparable states be in class IV, we have the solution (34)-(36). Then $\lambda_0 + \lambda_3 = \frac{1}{2}(1 + p_0 - p_4)$, which is less than or equals to $\frac{1}{2}$ when

$$p_4 \geq p_0. \quad (42)$$

We denote the region in A limited by (42) as A''' . In region A , we have $p_0 + p_3 > \frac{1}{2}$, from which we can derive $p_0 > p_{A'A''}$. Hence A' and A''' do not overlap. The region in A with $p_0 < p_4 < p_{A'A''}$ then is denoted as A'' . Hence the region A is divided into three layers from bottom to top as A', A'' and A''' along p_4 direction. The layers are shown in Fig.4 without considering parameter p_7 .

For layer A'' , the closest biseparable states neither belong to class I nor class IV. We consider class V. The solution then is (37)-(39). It is possible that A' and A''' may have class V closest biseparable states. However, we have $E_A \leq E_{A''}$ due to the convexity of the function $1 - H_2(x)$. We have $E_{A'''} \leq E_{A''}$ for $E_{A'''}$ is a part of $E_{A''}$. Thus closest biseparable states of A' and A''' can not be in class V.

We should further check if the closest biseparable states of A are in class III. We have proven that $E_A \leq E_B$ in last section, the proof is valid for layer A' . We only need to consider A'' and A''' . The closest biseparable states for layers A'' and A''' can not be class III (see appendix). The genuine REE is $E_A, E_{A''}, E_{A'''}$ for A', A'', A''' , respectively.

For all states in region A , inequality (12) is violated. In layer A' , inequalities (13) and (14) can be violated or not due to the location of the state. In layer A'' , (13) may or may not be violated due to the location of the state. In layer A''' , (13) is preserved since $1 \geq p_0 + p_4 + p_3 + p_7$ and $p_4 \geq p_0$. In both layers A'' and A''' , (14) is violated as shown in Fig.4.

Consider the closest biseparable states of class IV, we have $p_4 \geq p_0$ in order to preserve $\lambda_0 + \lambda_3 \leq \frac{1}{2}$. When (13) is violated, as in regions B and C , we have $2p_0 + p_3 + p_7 > 1 \geq p_0 + p_4 + p_3 + p_7$, the last inequality comes from the normalization condition. Thus $p_0 > p_4$. So regions B and C have not class IV solution. Class V is also not a solution for them (see appendix). The genuine REE keeps the same form as in the three parameter situation of last section for regions B and C . In regions B and C , inequality (13) is violated, in regions B and C_1 , inequality (12) is violated. Inequality (14) can either be violated or preserved in region B . In regions C and D_2 , the inequality (14) is preserved. Note that $p_0 + p_4 + p_3 + p_7 \leq 1$, we have $p_0 + 2p_3 + p_4 \leq 1 + p_3 - p_7$, thus if $p_3 \leq p_7$, we should have inequality (14). Region D_2 is biseparable even considering the fourth parameter p_4 .

Violation of inequality (14) is possible for regions D_1 as shown in Fig.4. Region D_1 is divided into two layers D'_1 and D''_1 , with D'_1 biseparable and D''_1 genuine entangled. Consider the class VI closest biseparable states for layer D'_1 , the genuine entanglement of layer D'_1 is $E_{A'''}$. Closest biseparable states in class I can not be the solution since D'_1 does not overlap with A' . Closest biseparable states in class V can not be the solution since $E_{A'''} \leq E_{A''}$. Class II solution requires $p_7 \geq p_3$, while in D'_1 we have $p_7 < p_3$.

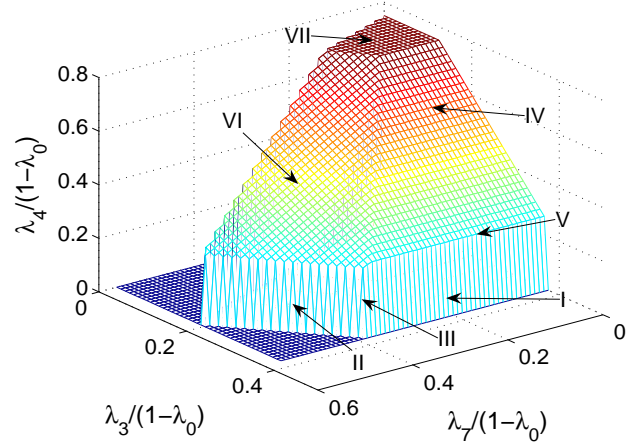


Figure 3: (Color online) Border surfaces of biseparable state set for $\lambda_0 = 0.2$. Surfaces I,II,IV are for $\lambda_0 + \lambda_3 = 1/2$, $2\lambda_0 + \lambda_3 + \lambda_7 = 1$, $\lambda_0 + 2\lambda_3 + \lambda_4 = 1$, respectively, they are the border surfaces of biseparable and genuine entangled states. Surfaces VI and VII are for $\lambda_0 + \lambda_3 + \lambda_4 + \lambda_7 = 1$ and $\lambda_4 = 3\lambda_0$, they and the surface $\lambda_7 = 3\lambda_3$ are the border surfaces of biseparable states and the unphysical region. Lines III and V are the intersections of the surfaces.

Class III is also not the solution for D'_1 (see appendix).

The situation of $p_4 > 3p_0$ does not exist due to the assuming of $p_0 = \max_{\beta,\gamma} F_{0\beta\gamma 0}$.

The parameter regions and the corresponding genuine REE are summarized in Table 1. All the regions should be subjected to the normalization constrain $p_4 + p_0 + p_3 + p_7 \leq 1$ and $p_4 \leq 3p_0$, $p_7 \leq 3p_3$.

6 Conclusions and Discussions

We have completely solved the genuine entanglement problem of four qubit cluster diagonal state. For any probability mixture of four qubit cluster basis states, we first reduce the total number of parameters from 15 to 8. Then we have proven that at most half of the necessary and sufficient biseparable criteria are violated for any genuine entangled state, the number of the parameters involved then is further reduced to 4. The four parameter states are classified as biseparable and genuine entangled. The entanglement measure for a genuine entangled state σ is the relative entropy of σ with respect to the closest biseparable state. We found that the closest biseparable state set is on the interface of biseparable state set and genuine entangled state set. The closest biseparable state set is divided into five classes. We classify the genuine entangled states into several regions and find their closest biseparable state classes. Each region has its closest biseparable state set. The entanglement is given analytically for any genuine entangled four qubit cluster diagonal state.

The four parameters are symmetry in some sense. For $p_0 + p_3 > \frac{1}{2}$, the parameter region is divided into five

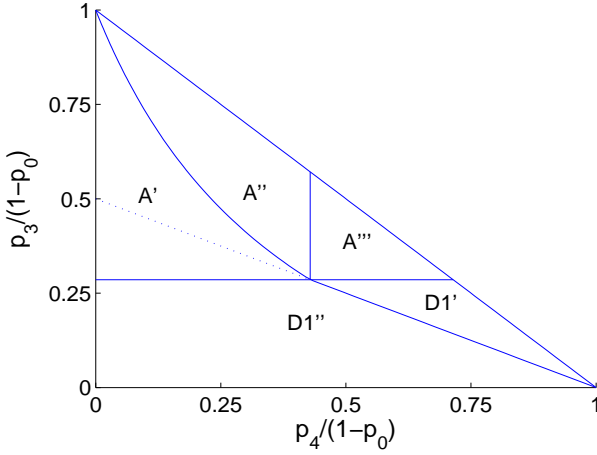


Figure 4: (Color online) Edge lines or curve of genuine entanglement regions for $p_0 = 0.3$. Genuine entanglement regions A' , A'' are based on the regions A_1 or A_2 , region A''' is based on A_2 in Fig.1, further characterized by the fourth parameter p_4 . The entanglement is $E_A, E_{A''}, E_{A'''}$, respectively. Biseparable region D_1'' and genuine entanglement region D_1' are based on region D_1 in Fig.1, further characterized by p_4 . The dash line is the border of inequality (14).

The genuine entangled or biseparable regions

	Descriptions	E	V
A'	$p_4 \leq p_{A'A''}; p_7 \leq p_{AB};$ $p_3 > \frac{1}{2} - p_0$	E_A	(12)(13)* (14)*
A''	$p_{A'A''} < p_4 < p_0; p_7 \leq p_{AB};$ $p_3 > \frac{1}{2} - p_0$	$E_{A''}$	(12)(13)* (14)
A'''	$p_4 \geq p_0; p_7 \leq p_{AB};$ $p_3 > \frac{1}{2} - p_0$	$E_{A'''}$	(12) (14)
B	$p_3 > p_7 > p_{AB}; p_3 > \frac{1}{2} - p_0$	E_B	(12)(13) (14)*
C_1	$p_7 \geq p_3; p_3 > \frac{1}{2} - p_0$	E_C	(12)(13)
C_2	$p_7 > 1 - 2p_0 - p_3;$ $p_3 \leq \frac{1}{2} - p_0$	E_C	(13)
D_1'	$p_4 > 1 - p_0 - 2p_3; p_7 < p_3;$ $p_3 \leq \frac{1}{2} - p_0$	$E_{A'''}$	(14)
D_1''	$p_4 \leq 1 - p_0 - 2p_3; p_7 < p_3;$ $p_3 \leq \frac{1}{2} - p_0$	0^*	None
D_2	$1 - 2p_0 - p_3 \geq p_7 \geq p_3;$ $p_3 \leq \frac{1}{2} - p_0$	0^*	None

Table 1: The regions classified by their parameters. Here E is the genuine entanglement. V means violation of inequalities (12)(13) and (14). (13)* or (14)* represents that (13) or (14) is violated in some part of the region and preserved in the other part of the region. 0^* is 0 only when (15)-(17) are preserved.

subregions: A''' , A'' , A' , B and C_1 , where C_1 and A''' are symmetric, B and A'' are symmetric, A' is self-symmetric under the exchange

$$(p_0, p_4) \Leftrightarrow (p_3, p_7). \quad (43)$$

For $p_0 + p_3 \leq \frac{1}{2}$, the parameter region is divided into four subregions: D_1' , D_1'' , D_2 and C_2 . The regions D_1' and C_2 are symmetric with each other in the sense of the (43). D_1'' and D_2 are the biseparable subregions in the four parameter system. Notice that when all of the inequalities (12)-(14) are preserved, we should consider the violation of inequalities (15)-(17). A similar analysis should be added for the parameters p_1, p_2, p_5, p_6 . The actual case is that D_1' and D_2 should be further divided into truly biseparable subregions and genuine entangled subregions according to the parameters p_1, p_2, p_5, p_6 .

The genuine relative entropy of entanglement is analytically expressed as five formulas, according to the subregion the genuine entangled state belongs to. The five formulas can be further classified as three kinds: one symmetric formula and two pairs. The pair formulas can be interchangeable under the parameter exchange (43).

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Appendix: Comparison of E_B and $E_{A''}$

Define a function

$$E(x) = 1 - xH_2\left(\frac{p_3}{x}\right) - (1-x)H_2\left(\frac{p_0}{1-x}\right),$$

then E_B and $E_{A''}$ can be expressed as $E_B = E(x_B)$, $E_{A''} = E(x_{A''})$ with $x_B = p_3 + p_7$, $x_{A''} = 1 - p_0 - p_4$. Since $p_3 + p_7 + p_0 + p_4 \leq 1$, thus

$$x_B \leq x_{A''}.$$

The derivative of the function $E(x)$ is

$$\frac{dE(x)}{dx} = -\log_2 \frac{x}{1-x} + \log_2 \frac{x-p_3}{1-x-p_0},$$

which leads to the solely extremal point

$$x^* = \frac{p_3}{p_0 + p_3}.$$

At the extremal point x^* , the function $E(x)$ reaches its minimum $E(x^*)$ since the second derivative at $x = x^*$ is positive. The function $E(x)$ monotonically decreases with x for $x \leq x^*$, $E(x)$ monotonically increases with x for $x \geq x^*$.

Let the genuine entangled state be at regions B, C , we have $p_7 > p_{AB}$. We can rewrite it as $p_3 + p_7 > \frac{p_3}{p_0 + p_3}$.

Hence $x^* < x_B \leq x_{A''}$. At the right side of x^* , the function $E(x)$ is a monotonically increasing function, so that

$$E_B \leq E_{A''}.$$

In region B , the genuine entanglement is E_B . In region C , the genuine entanglement is E_C since $E_C \leq E_B \leq E_{A''}$.

Consider the regions A'' , A''' and D'_1 , we have $p_4 > p_{A'A''}$. We rewrite it as $p_0 + p_4 > \frac{p_0}{p_0 + p_3}$, further we have $1 - p_0 - p_4 < \frac{p_3}{p_0 + p_3}$, which is $x_{A''} < x^*$. Hence $x_B \leq x_{A''} < x^*$. At the left side of x^* , the function $E(x)$ is a monotonically decreasing function, so that

$$E_B \geq E_{A''}.$$

In regions A'' , the genuine entanglement is $E_{A''}$. In regions A''' and D'_1 , the genuine entanglement is $E_{A'''}$ since $E_{A'''} \leq E_{A''} \leq E_B$.

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